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Moduli spaces of maximally supersymmetric solutions on non-commutative tori and non-commutative orbifolds

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Abstract: A maximally supersymmetric configuration of super Yang-Mills living on a non-commutative torus corresponds to a constant curvature connection. On a non-commutative toroidal orbifold there is an additional constraint that the connection be equivariant. We study moduli spaces of (equivariant) constant curvature connections on non-commutative even-dimensional tori and on toroidal orbifolds. As an illustration we work out the cases of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds in detail. The results we obtain agree with a commutative picture describing systems of branes wrapped on cycles of the torus and branes stuck at exceptional orbifold points.

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Contents

1. Introduction

The idea that String theory leads to some sort of fuzzy or non-commutative microscopic structure of space-time has been around for quite a while. A new boost to this idea emerged after the paper [3]. It was shown in that paper that non-commutative tori arise as particular compactifications of M(atrix) theory ([1]) that is conjectured to be a non-perturbative definition of String theory (see [4]–[15] and references therein for the subsequent development). Later in [16] a number of important results concerning relations between String theory and non-commutative geometry were obtained. In particular the conditions under which non-commutative geometry arises within perturbative open string theory were clarified.

On the mathematics side non-commutative tori are the best studied examples of non-commutative spaces (see [23] for a good overview). An important notion in non-commutative geometry is that of Morita equivalence that gives some equivalence relation between algebras of functions on non-commutative spaces. A striking result first observed in [3] and later on proved rigorously in [5] is that non-commutative world volume field theories living on non-commutative tori are invariant under duality transformations generated by Morita equivalences of non-commutative tori. This duality is directly related to T-duality of perturbative string theory compactifications ([16, 25]).

It seems to be interesting and important to study compactifications on other non-commutative spaces and see how much of non-commutative geometry techniques that proved to be useful for tori can be extended (see [6, 7, 21, 22] for some work done in that direction). In paper [21] we studied M(atrix) theory compactifications on non-commutative toroidal orbifolds. That paper primarily concentrated on two-dimensional $\mathbb{Z}_2$ orbifolds. In the present paper we continue investigation of M(atrix) theory compactification on non-commutative spaces. The main topic of this paper is the structure of projective modules over non-commutative orbifolds that admit a constant curvature Yang-Mills field and moduli spaces of all such fields. Such modules and constant curvature connections on them describe configurations preserving half of the unbroken supersymmetries. All our results concern classical aspects only. However counting of quantum states with specified brane charges can be made in terms of a supersymmetric sigma model on the appropriate classical moduli space provided we have a sufficient number of supersymmetries.

In the commutative case configurations with vanishing SU(N) part of the curvature and their interpretations in terms of D-branes were considered in a number of papers (see [17] and references therein). D0-branes on toroidal $\mathbb{Z}_2$ orbifolds were studied in [18, 19]. Whenever our results can be compared with the commutative results we observe a complete agreement.

The paper is organized as follows. Section 2 contains some generalities on non-commutative tori and matrix theory as well as an explanation of our general strategy regarding the moduli space problem. In essence the novelty of our approach is in the following. In the usual approach one first fixes a module, i.e. a representation of algebra of functions on a non-commutative space, and then considers all constant curvature connections modulo gauge transformations. Instead we fix the connection ∇_i and then look at all equivalence classes of the torus representations compatible with that connection. If $[\nabla_i, \nabla_j] = f_{ij} \cdot \mathbf{1}$ where f_{ij} is a non-degenerate matrix then the algebra generated by ∇_i is isomorphic to Heisenberg algebra. We can use then the well known results about complete reducibility of Heisenberg algebra representations. The case of generic constant curvature connection can be reduced to the non-degenerate case by use of Morita equivalence provided the torus dimension is even. The Morita equivalence technique by itself is not directly relevant to the present paper; its discussion is relegated to the appendix. We restrict ourselves to the even-dimensional case in this paper. However after appropriate modification our method also works for odd dimensions.

In section 3 we apply our general strategy to non-commutative tori and prove that the moduli space is isomorphic to $(\tilde{\mathbb{T}}^d)^r/S_r$ where $\tilde{\mathbb{T}}^d$ is a commutative d-dimensional torus and r is the greatest common divisor of topological integers (D-brane charges) characterizing the module. For $d = 2$ this result was proved by A. Connes and M. Rieffel in [24].

In section 4 we introduce non-commutative toroidal orbifolds and outline how the approach used for tori can be extended to the study of moduli space of equivariant constant curvature connections.

In sections 5 and 6 we study in detail $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds, respectively. For these cases we work out a general construction of a module that admits a constant curvature equivariant connection. In each case a module is built out of standard blocks that can be interpreted to describe D0 particles and various membranes stuck at exceptional points of the orbifold. The corresponding moduli spaces are proved to be isomorphic to $(\tilde{\mathbb{T}}^d/\mathbb{Z}_2)^m/S_m$ and $(\tilde{\mathbb{T}}^d/\mathbb{Z}_4)^n/S_n$, respectively. Here m and n are some integers that depend on topological numbers of the module. In the two-dimensional case $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds of non-commutative tori were studied by S. Walters in a number of papers ([28]). It would be interesting to calculate the topological numbers introduced in those papers for the modules we consider.

In section 7 we add scalar fields to the discussion and describe how Coulomb branches vary over the moduli space of constant curvature connections.

Finally the appendix contains a general discussion of Morita equivalence for toroidal orbifolds and details of its application to $\mathbb{Z}_2$ and $\mathbb{Z}_4$ cases.

2. 1/2 BPS configurations on non-commutative tori

We start this section by discussing some general aspects of constant curvature connections on non-commutative tori.

We define an algebra $\mathbb{T}_\theta$ of smooth functions on a d-dimensional non-commutative torus in the following way. Let $D \subset \mathbb{R}^d$ be a lattice $D \cong \mathbb{Z}^d$ and θ_{ij} be an antisymmetric $d\times d$ matrix. The algebra $\mathbb{T}_\theta$ is an associative algebra whose elements are formal series

$$\sum_{\mathbf{n}\in D} C(\mathbf{n})U_{\mathbf{n}}\,,$$

where $C(\mathbf{n})$ are complex numbers and $U_{\mathbf{n}}$ satisfy the relations

$$U_{\mathbf{n}}U_{\mathbf{m}} = U_{\mathbf{n}+\mathbf{m}}e^{\pi i n_j \theta^{jk} m_k}\,. \tag{2.1}$$

Here the coefficients $C(\mathbf{n})$ are assumed to decrease faster than any power of $|\mathbf{n}|$ as $\mathbf{n}\to\infty$. We will denote generators corresponding to standard basis vectors $(n^i)_j = \delta^i_j$ by U_i. Any $U_{\mathbf{n}}$ can be expressed as a product of U_i's times a numerical phase factor.

The notion of a vector bundle (or rather that of a space of sections of a vector bundle) in non-commutative geometry is replaced by a projective module over $\mathbb{T}_\theta$. Unless specified we will assumed that we work with a right module (i.e. we have a right action of $\mathbb{T}_\theta$). Let L be a d-dimensional commutative Lie algebra acting on $\mathbb{T}_\theta$ by means of derivations

$$\delta_j U_{\mathbf{n}} = 2\pi i n_j U_{\mathbf{n}}\,. \tag{2.2}$$

A connection on a projective module E over $\mathbb{T}_\theta$ is defined in terms of operators $\nabla_i : E \to E$ satisfying a Leibniz rule

$$\nabla_i(ea) = (\nabla_i e)a + e(\delta_i a)$$

for any $e \in E$ and any $a \in \mathbb{T}_\theta$.

It was first shown in ([3]) how (super)Yang-Mills theory on a non-commutative torus arises as a particular compactification of M(atrix) theory ([1, 2]). The Minkowski action functional of M(atrix) theory compactified on a non-commutative torus $\mathbb{T}_\theta$ can be written as

$$\begin{aligned} S = \frac{-V}{4g_{\mathrm{YM}}^2}\,\mathrm{Tr}\Big((F_{ij} + \phi_{ij}\mathbf{1})^2 + [\nabla_i, X_I]^2 + [X_I, X_J]^2\Big) + \\ + \frac{iV}{2g_{\mathrm{YM}}^2}\,\mathrm{Tr}\Big(\bar\psi\Gamma^i[\nabla_i, \psi] + \bar\psi\Gamma^I[X_I, \psi]\Big)\,, \end{aligned} \tag{2.3}$$

where ∇_i, $i = 1, \ldots, d$ is a connection on a $\mathbb{T}_\theta$-module E, ψ is a ten-dimensional Majorana-Weyl spinor taking values in the algebra $\mathrm{End}_{\mathbb{T}_\theta} E$ of endomorphisms of E, ϕ_{ij} is an antisymmetric tensor, $X_I \in \mathrm{End}_{\mathbb{T}_\theta} E$, $I = d+1, \ldots 10$ are scalar fields, Γ_μ are ten-dimensional gamma-matrices.

The action (2.3) is invariant under the following supersymmetry transformations:

$$\begin{aligned} \delta\nabla_i = \frac{i}{2}\bar\epsilon\Gamma_i\psi\,, \qquad \delta X_I = \frac{i}{2}\bar\epsilon\Gamma_I\psi\,, \\ \delta\psi = \tilde\epsilon - \frac{1}{4}\Big(F_{ij}\Gamma^{ij} + 2[\nabla_i, X_I]\Gamma^{iI} + [X_I, X_J]\Gamma^{IJ}\Big)\epsilon\,, \end{aligned} \tag{2.4}$$

where ϵ, $\tilde\epsilon$ are 10D Majorana-Weyl spinors parameterizing the transformation.

Classical configurations preserving 1/2 of these supersymmetries have a vanishing field ψ and the fields ∇_i, X_I satisfying the equations

$$[\nabla_j, \nabla_k] = 2\pi i f_{jk}\cdot\mathbf{1}\,, \qquad [\nabla_j, X_I] = 0\,, \qquad [X_I, X_J] = 0\,, \tag{2.5}$$

where f_{ij} are constants and $\mathbf{1}$ is a unit endomorphism. We call solutions to the first equation modulo gauge transformations a Higgs branch of the 1/2-BPS moduli space. The whole moduli space of solutions to (2.5) can be viewed as a fibration over the Higgs branch. We will first describe the Higgs branch which is the moduli space of constant curvature connections and then take into account the scalar fields.

Let us outline here the strategy we will take addressing the moduli space problem. The complete set of equations that defines a module over a non-commutative torus and a constant curvature connection on it reads

$$U_jU_k = e^{2\pi i\theta_{jk}}U_kU_j\,, \qquad [\nabla_j, U_k] = 2\pi i\delta_{jk}U_k\,, \qquad [\nabla_j, \nabla_k] = 2\pi i f_{jk}\,. \tag{2.6}$$

If U_i, ∇_j are operators in Hilbert space E satisfying these equations and $Z : E \to E$ is a unitary linear (over $\mathbf{C}$) operator, then ZU_iZ^{-1}, $Z\nabla_iZ^{-1}$ also solve (2.6). The usual approach to eqs. (2.6) is to fix generators U_i, i.e. fix a module, and then look for ∇_i satisfying the last two equations in (2.6). The transformations $Z : E \to E$ preserving U_i's are unitary endomorphisms. We will take a different approach in this paper and fix the representation of ∇_i. The gauge transformations in this picture are given by unitary operators Z that commute with all ∇_i. The moduli space is then a space of solutions U_i to the first two equations in (2.6) modulo gauge transformations. Note that in this approach inequivalent modules, that in our case are modules that have distinct Chern (and/or other topological numbers in the case of orbifolds), are treated simultaneously. Fixing topological numbers then corresponds to choosing a connected component of the total moduli space of solutions to (2.6).

The (super)Yang-Mills theories (2.3) on Morita equivalent non-commutative tori $\mathbb{T}_\theta$, $\mathbb{T}_{\hat{\theta}}$ are physically equivalent (see [5]). To any module E over $\mathbb{T}_\theta$ there corresponds a module $\hat{E}$ over $\mathbb{T}_{\hat{\theta}}$ and to any connection ∇ on E there corresponds a connection $\hat{\nabla}$ and vice versa. The correspondence of connections respects gauge equivalence relation and maps a constant curvature connection into a constant curvature one. This means that moduli spaces of constant curvature connections are isomorphic for Morita equivalent tori. A crucial fact that we are going to exploit analyzing representations of (2.6) is that by use of Morita equivalence one can reduce the problem to the situation when f_{ij} is a non-degenerate matrix (see appendix A).

3. Constant curvature connections on non-commutative tori

Let $\mathbb{T}_\theta$ be a $d = 2g$-dimensional non-commutative torus. Let E be a projective module over $\mathbb{T}_\theta$ admitting a constant curvature connection ∇_i, $i = 1, \ldots, d$

$$[\nabla_j, \nabla_k] = 2\pi i f_{jk} \cdot \mathbf{1}\,, \tag{3.1}$$

where f_{jk} is a constant non-degenerate anti-symmetric matrix with real entries. (We assume that ∇_i are antihermitian operators.) Up to normalizations algebra (3.1) is the Heisenberg algebra specified by canonical commutation relations for g degrees of freedom. According to the Stone-von Neumann theorem there is a unique irreducible representation $\mathcal{F}$ of algebra (3.1). We assume that E can be decomposed into a direct sum of a finite number of irreducible components: $E \cong \mathcal{F}^N \cong \mathcal{F} \otimes \mathbb{C}^N$.

We fix the representation $\mathcal{F}$ as follows. First let us bring the matrix f_{ij} to a canonical block-diagonal form

$$(f_{ij}) = \begin{pmatrix} f_1\epsilon & 0 & \ldots & 0 \\ 0 & f_2\epsilon & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & f_g\epsilon \end{pmatrix}, \tag{3.2}$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.3}$$

is a 2×2 matrix and f_i are positive numbers.

Then we can define a representation space as $L_2(\mathbb{R}^g)$ and the operators ∇_i as

$$\nabla_j = \sqrt{f_{(j+1)/2}}\partial_j\,, \quad j-\text{odd}\,, \quad \nabla_j = i\sqrt{f_{j/2}}x_{j-1}\,, \quad j-\text{even}\,, \tag{3.4}$$

where ∂_j, x_k, $j,k = 1,\ldots,g$ are derivative and multiplication by x^k operators acting on smooth functions $f(x) \in L_2(\mathbb{R}^g)$. An arbitrary representation of the torus generators U_i, $i = 1,\ldots,d$ has the form $U_i = U_i^{\text{st}} \cdot u_i$ where U_i^{st} is some standard representation satisfying

$$[\nabla_j, U_k^{\text{st}}] = 2\pi i \delta_{jk} U_k^{\text{st}}$$

and u_i is an $N \times N$ unitary matrix. This form of representation of U_i follows from the irreducibility of representation $\mathcal{F}$. A straightforward calculation shows that one can take U_i^{st} to be

$$U_k^{\text{st}} = e^{-(f^{-1})^{kl}\nabla_l}\,. \tag{3.5}$$

These operators satisfy

$$U_j^{\text{st}}U_k^{\text{st}} = e^{-2\pi i (f^{-1})^{jk}} U_k^{\text{st}}U_j^{\text{st}}\,. \tag{3.6}$$

Since $U_i = U_i^{\text{st}} \cdot u_i$ must give a representation of a non-commutative torus it follows from (3.6) that so must do the operators u_i. But the last ones are finite-dimensional matrices so they can only represent a non-commutative torus whose noncommutativity matrix has rational entries, i.e. u_i's have to satisfy

$$u_i u_j = e^{2\pi i n^{ij}/N} u_j u_i\,, \tag{3.7}$$

where N is a positive integer and n^{ij} is an integer valued antisymmetric matrix. Putting the formulas (3.6) and (3.7) together one finds that U_i's give a representation of a non-commutative torus $\mathbb{T}_\theta$ with

$$\theta^{ij} = -(f^{-1})^{ij} + n^{ij}/N\,. \tag{3.8}$$

It follows from the results obtained by M. Rieffel ([27]) that for finite N (i.e. when E decomposes into a finite number of irreducible components) the module E endowed with $U_i = U_i^{\text{st}} \cdot u_i$ as above is a finitely generated projective module over $\mathbb{T}_\theta$ with θ given in (3.8). Conversely one can show that the finiteness of N is required by the condition of E to be finitely generated and projective. (See [26] for a detailed discussion of modules admitting a constant curvature connection.)

Topological numbers. Let us calculate here the topological numbers of the modules constructed above. We assume here that the matrix θ_{ij} given in (3.8) has irrational entries. Then a projective module E is uniquely characterized by an integral element $\mu(E)$ of the even part of Grassmann algebra $\Lambda^{\text{even}}(\mathbb{R}^d)$. In order to calculate $\mu(E)$ we can use Elliot's formula

$$\mu(E) = \exp\left(\frac{1}{2}\frac{\partial}{\partial \alpha^i}\theta^{ij}\frac{\partial}{\partial \alpha^j}\right)\text{ch}(E)\,, \tag{3.9}$$

where

$$\mathrm{ch}(E) = D \cdot \exp\left(\frac{1}{2\pi i}\alpha^i f_{ij}\alpha^j\right) \tag{3.10}$$

and D is a non-negative real number that plays a role of the dimension of a module in non-commutative geometry. From (3.9) applying a Fourier transform we obtain

$$\begin{aligned}\mu(E) &= D \cdot \mathrm{Pfaff}(f) \cdot \int d\beta\, \exp\left(\frac{1}{2}\beta_i\left((f^{-1})^{ij} + \theta^{ij}\right)\beta_j + \alpha^i\beta_i\right) \\ &= D \cdot \mathrm{Pfaff}(f) \cdot \int d\beta\, \exp\left(\frac{1}{2}\beta_i n^{ij}\beta_j/N + \alpha^i\beta_i\right). \end{aligned}\tag{3.11}$$

At this point it is convenient to assume that the matrix n_{ij} is brought to a canonical block-diagonal form similar to (3.2) with integers n_i, $i = 1,\ldots,g$ on the diagonal by means of an $\mathrm{SL}(d,\mathbb{Z})$ transformation (this is always possible, see [30]). Then we can explicitly do the integration in (3.11) and obtain

$$\mu(E) = C \cdot \left(\frac{n_1}{N} + \alpha^1\alpha^2\right)\left(\frac{n_2}{N} + \alpha^3\alpha^4\right) \cdot \cdots \cdot \left(\frac{n_g}{N} + \alpha^{2g-1}\alpha^{2g}\right), \tag{3.12}$$

where $C = D \cdot \mathrm{Pfaff}(f)$ is a constant that can be determined by the requirement that $\mu(E)$ is an integral element of Grassmann algebra Λ (i.e. each coefficient is an integer). By looking at the term with a maximal number of α's in (3.12) we immediately realize that C must be an integer. We will prove below that $C = N$.

Let us introduce numbers

$$N_i = \frac{N}{\mathrm{g.c.d.}(N, n_i)}, \qquad \tilde{N}_i = \frac{n_i}{\mathrm{g.c.d.}(N, n_i)}$$

so that for each $i = 1,\ldots,g$ the pair N_i, $\tilde{N}_i$ is relatively prime. Then we can rewrite (3.12) as

$$\mu(E) = \frac{C}{N_1N_2 \cdot \cdots \cdot N_g}\prod_{i=1}^{g}(\tilde{N}_i + N_i\alpha^{2i-1}\alpha^{2i}) \equiv \frac{C}{N_1N_2 \cdot \cdots \cdot N_g}\mu_0(E). \tag{3.13}$$

For any integral element ν of Grassmann algebra Λ let us introduce a number $\mathrm{g.c.d.}(\nu)$ which is defined to be the largest integer k such that $\nu = k\nu'$ where ν' is also integral. It is a simple task to prove by induction in g that $\mathrm{g.c.d.}(\mu_0(E)) = 1$. Hence, C must be an integer divisible by the product $N_1N_2 \cdot \cdots \cdot N_g$. Moreover $C = \mathrm{g.c.d.}(\mu(E))N_1N_2 \cdot \cdots \cdot N_g$. It is known (for example see [29]) that the dimension of an irreducible representation of the algebra (3.7) is equal to the product $N_1 \cdot N_2 \cdot \cdots \cdot N_g$. Thus, necessarily this product divides N, i.e. $N = N_1 \cdot N_2 \cdot \cdots \cdot N_g \cdot N_0$ where N_0 is an integer equal to the number of irreducible components in the representation $\mathbb{C}^N$ of the algebra (3.7). Evidently N_0 divides $\mathrm{g.c.d.}(\mu(E))$. We are going to show below that $\mathrm{g.c.d.}(\mu(E))$ cannot be bigger than N_0. This will prove that $C = N$.

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Let us look at some particular examples of formula (3.12). If the matrix n^{ij} is non degenerate then $\mu(E)$ is a quadratic exponent:

$$\mu(E) = p \cdot \exp\left(\frac{1}{2}\alpha^i (n^{-1})_{ij} \alpha^j N\right), \quad p = N_0 \tilde{N}_1 \cdot \dots \cdot \tilde{N}_g\,, \tag{3.14}$$

where $p = N \cdot \mathrm{Pfaff}(n/N)$ is written in a form where it is manifestly an integer. If n^{ij} is degenerate then $\mu(E)$ is a so called generalized quadratic exponent (see [26] and [11], Appendix D). For example if $n^{ij} = 0$ for all i and j then we obtain from (3.11)

$$\mu(E) = N\alpha^1\alpha^2 \cdot \dots \cdot \alpha^d\,. \tag{3.15}$$

Moduli space of constant curvature connections in terms of irreps of rational n.c. tori. In the previous subsection we showed that modules endowed with a constant curvature connection correspond to representations of matrix algebra (3.7). The residual gauge transformations preserving (3.4) correspond to $N \times N$ unitary transformations acting on the $\mathbb{C}^N$ factor of E. Thus, we see that the moduli space of constant curvature connections on a module with fixed (N, n^{ij}) (or fixed $\mu(E)$, which is the same) can be described as a space of inequivalent representations of the matrix algebra (3.7). The center of algebra (3.7) is spanned by elements $u_{\mathbf{k}}$ with $\mathbf{k} \in D \cong \mathbf{Z}^d$ satisfying $k_i n^{ij} m_j / N \in \mathbb{Z}$ for any $\mathbf{m} \in D$. Such elements correspond to a sublattice of D that we denote $\tilde{D}$. To describe the center in a more explicit way it is convenient to choose the basis we used above, in which the matrix n^{ij} is brought to a block-diagonal canonical form. In this basis generators of the center can be chosen to be elements $(u_i)^{M_i}$ where we set $M_i = N_{(i+1)/2}$, i-odd and $M_i = N_{i/2}$, i-even. Thus, in an irreducible representation $(u_i)^{M_i} = \lambda_i \in \mathbb{C}^{\times}$ are constants of absolute value 1.

Using the substitution

$$u_i \longmapsto c_i u_i\,, \tag{3.16}$$

where c_i are constants, $|c_i| = 1$, we obtain an irreducible representation with values of center generators $\lambda_i' = \lambda_i c_i^{M_i}$. By means of this substitution one can transform any irrep into the one with $\lambda_i = 1$. The last one corresponds to a representation of the algebra specified by relations (3.7) along with the relations $(u_i)^{M_i} = 1$. This algebra has a unique irreducible representation of dimension $N_1 \cdot N_2 \cdot \dots \cdot N_g$ (for example see [29]). Therefore, the space of irreducible representations of algebra (3.7) is described by means of d complex numbers λ_i with absolute value 1, i.e. is isomorphic to a (commutative) torus $\tilde{\mathbb{T}}^d \cong \mathbb{R}^{*d}/\tilde{D}^*$. We denote the corresponding irreps by E_Λ, $\Lambda = (\lambda_1, \dots, \lambda_d)$. In general for any non-commutative torus $\mathbb{T}_\theta$ one can construct a group L_θ of automorphisms isomorphic to a commutative torus of the same dimension by means of (3.16). This torus acts naturally on the space of unitary representations of $\mathbb{T}_\theta$. If θ is rational we obtain a transitive action of this automorphism group on the space of irreducible representations. In this case one can consider L_θ as a finite covering of $\tilde{\mathbb{T}}^d$.

Let us assume now that the space $\mathbb{C}^N$ is decomposed into irreducible representations of algebra (3.7)

$$\mathbb{C}^N = E_{\Lambda_1} \oplus \cdots \oplus E_{\Lambda_{N_0}} \,. \tag{3.17}$$

Note that in the picture we are working with, gauge transformations are given by unitary linear operators acting on E that commute with all ∇_i's that is by unitary $N \times N$ matrices. The matrices representing central elements are diagonalized in the basis specified by decomposition (3.17). There are residual gauge transformations corresponding to permutations of diagonal entries. Thus, we see that in general the moduli space is isomorphic to $(\tilde{\mathbb{T}}^d)^{N_0}/S_{N_0}$. As it was noted in the previous subsection N_0 divides g.c.d.$(\mu(E))$. On the other hand as we know from [26, 11] any module E over a non-commutative torus $\mathbb{T}_\theta$ admitting a constant curvature connection ∇_i can be represented as a direct sum of k identical modules $E = E' \oplus \cdots \oplus E'$ with $k =$ g.c.d.$(\mu(E))$. This implies that the moduli space of constant curvature connections necessarily contains a subset isomorphic to $(\tilde{\mathbb{T}}^d)^k/S_k$. Thus, on dimensional grounds we conclude that $k =$ g.c.d.$(\mu(E)) = N_0$.

4. General toroidal orbifolds

In this section we will consider how the considerations above can be generalized to the case of toroidal orbifolds. We will work out in detail the particular cases of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds in the subsequent sections. Let $D \subset \mathbb{R}^d$ be a d-dimensional lattice embedded in $\mathbb{R}^d$ and let G be a finite group acting on $\mathbb{R}^d$ by linear transformations mapping the lattice D to itself. For an element $g \in G$ we will denote the corresponding representation matrix $R_i^j(g)$. One can write down constraints describing compactification of M(atrix) theory on the orbifold $\mathbb{T}^d/G$, where $\mathbb{T}^d = \mathbb{R}^d/D$:

$$X_j + \delta_{ij} 2\pi \cdot \mathbf{1} = U_i^{-1} X_j U_i \,, \tag{4.1}$$

$$X_I = U_i^{-1} X_I U_i \,, \qquad \psi_\alpha = U_i^{-1} \psi_\alpha U_i \,, \tag{4.2}$$

$$R_i^j(g) X_j = W^{-1}(g) X_i W(g) \,, \tag{4.3}$$

$$\Lambda_{\alpha\beta}(g) \psi_\beta = W^{-1}(g) \psi_\alpha W(g) \,, \qquad X_I = W^{-1}(g) X_I W(g) \,. \tag{4.4}$$

Here $i, j = 1, \ldots, d$ are indices for directions along the torus, $I = d+1, \ldots, 9$ is an index corresponding to the transverse directions, α is a spinor index; $\Lambda_{\alpha\beta}(g)$ is the matrix of spinor representation of G obeying $\Lambda^\dagger(g)\Gamma^i\Lambda(g) = R_{ij}(g)\Gamma_j$; U_i, $W(g)$ - unitary operators. One can check that the quantities $U_i U_j U_i^{-1} U_j^{-1}$ commute with all X_i, X_I, and ψ_α. It is natural to set them to be proportional to the identity operator. This gives us defining relations of a non-commutative torus

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j \,.$$

It is convenient to work with linear generators $U_{\mathbf{n}}$ that can be expressed in terms of products of U_i. One can further check that expressions $W(gh)W^{-1}(g)W^{-1}(h)$ and

$W^{-1}(g)U_{\mathbf{n}}W(g)U^{-1}_{R^{-1}(g)\mathbf{n}}$ also commute with all fields X_i, X_I, ψ_α. We assume that these expressions are proportional to the identity operator. This leads us to the following relations:

$$\begin{aligned} W(g)W(h) &= W(gh)e^{i\phi(g,h)}\,, \\ W^{-1}(g)U_{\mathbf{n}}W(g) &= U_{R^{-1}(g)\mathbf{n}}e^{i\chi(\mathbf{n},g)}\,, \end{aligned} \tag{4.5}$$

where $\phi(g,h)$, $\chi(\mathbf{n},g)$ are constants. The first equation means that operators $W(g)$ furnish a projective representation of G. It follows from these equations that the matrix θ is invariant under the group action $R(g)$. Below we will confine ourselves to the case of vanishing cocycles ϕ and χ. We refer the reader to papers [6, 20] for a discussion of cases when cocycle ϕ does not vanish. (Note that for cyclic groups both cocycles are always trivial. This means that they can be absorbed into redefinitions of generators.)

One can define an algebra of functions on a non-commutative orbifold as an algebra generated by the operators $U_{\mathbf{n}}$ and $W(g)$ satisfying (2.1) and

$$W^{-1}(g)U_{\mathbf{n}}W(g) = U_{R^{-1}(g)\mathbf{n}}\,, \qquad W(g)W(h) = W(gh)\,. \tag{4.6}$$

These equations mean that the algebra at hand is a crossed product $\mathbb{T}_\theta \rtimes_R G$. Again we remark here that allowing central extensions gives a more general case of twisted crossed products. In this paper we will concentrate on the untwisted case.

The algebra $\mathbb{T}_\theta \rtimes_R G$ can be equipped with an involution $*$ by setting $U^*_{\mathbf{n}} = U_{-\mathbf{n}}$, $W^*(g) = W(g)$. This makes it possible to embed these algebras into a general theory of C^* algebras. A projective module over an orbifold can be considered as a projective module E over $\mathbb{T}_\theta$ equipped with operators $W(g)$, $g \in G$ satisfying (4.6). Equations (4.1) and (4.3) mean that X_i specifies a G- equivariant connection on E, i.e. ∇_i is a $\mathbb{T}_\theta$ connection satisfying

$$R^j_i(g)\nabla_j = W^{-1}(g)\nabla_i W(g)\,. \tag{4.7}$$

The fields X_I are endomorphisms of E, commuting both with $U_{\mathbf{n}}$ and $W(g)$ and the spinor fields ψ_α can be called equivariant spinors.

Let us comment here on the supersymmetry of these compactifications. The surviving supersymmetry transformations are transformations (2.4) corresponding to invariant spinors ϵ, $\tilde{\epsilon}$, i.e. the ones satisfying $\Lambda(g)\epsilon = \epsilon$. For $d = 4$, 6 this equation has a non-trivial solution provided the representation $R(g)$ lies within an SU(2) or SU(4) subgroup, respectively. The possible finite groups G that can be embedded in this way are well known. Those include the examples of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ four-dimensional orbifolds to be considered below. In those cases when the supersymmetry is not broken completely equivariant connections of constant curvature correspond to configurations preserving half of the unbroken supersymmetries.

Now we restrict ourselves to modules admitting a constant curvature equivariant connection (3.1) and (4.7). All the steps of the analysis made above for the case of tori leading to the decomposition (3.17) can be repeated in a straightforward way. We should add now to that analysis operators $W(g)$. Equation (4.7) implies that the curvature tensor f_{ij} is invariant under the action of G. As above we fix a representation of the Heisenberg algebra (3.1) in the form (3.4). Then a connection $\nabla_i^g = R_i^j(g)\nabla_j$ gives a representation of the same Heisenberg algebra (3.1). By uniqueness of irreducible representation $\mathcal{F}$ there exists a set of operators $W^{\rm st}(g)$: $\mathcal{F} \to \mathcal{F}$ satisfying (4.7). It follows from the definition (3.5) of $U_j^{\rm st}$ that $W^{\rm st}(g)$ and $U_j^{\rm st}$ commute as in (4.6). This implies that a general set of operators $W(g) : E \to E$ has a form

$$W(g) = W^{\rm st}(g) \otimes w(g)\,, \tag{4.8}$$

where $w(g)$ are $N \times N$ matrices satisfying

$$w^{-1}(g) u_{\mathbf{n}} w(g) = u_{R^{-1}(g)\mathbf{n}}\,. \tag{4.9}$$

Here $u_{\mathbf{n}}$ are (linear) generators of the rational torus (3.7). In general relation (4.7) only implies that the operators $W^{\rm st}(g)$ form a projective representation of G. Then, the commutation relations for $w(g)$ are twisted by an opposite cocycle. The problem of describing moduli space of equivariant constant curvature connections now boils down to the study of irreducible representations of the matrix algebra generated by u_i, $w(g)$.

As it was explained above an irreducible representation of a rational torus is labeled by a point of the commutative torus $\tilde{\mathbb{T}}^d = \mathbb{R}^{*d}/\tilde{D}^*$. The sublattice $\tilde{D}$ defining the center of (3.7) is preserved by G and so is its dual $\tilde{D}^*$. Therefore, G acts on the torus $\tilde{\mathbb{T}}^d$. We denote this action by $R^*(g)$. It follows from (4.9) that an irreducible representation E_Λ from decomposition (3.17) is mapped by $w(g)$ into $E_{R^*(g)\Lambda}$. Therefore, the set of Λ_i in the decomposition (3.17) has to be invariant under the action $R^*(g)$. The irreducible representations of the algebra generated by u_i, $w(g)$ can be grouped according to the orbits of G action on $\tilde{\mathbb{T}}^d$. For a generic point one has an orbit of length $|G|$. There are also exceptional points that include fixed points and points whose orbits length is a non-trivial divisor of $|G|$. If the exceptional points are isolated the corresponding representations can be interpreted in terms of branes that are stuck at the exceptional points. Below we will consider in detail the structure of the aforementioned representations for the orbifold groups $\mathbb{Z}_2$ and $\mathbb{Z}_4$.

5. Non-commutative $\mathbb{Z}_2$ orbifolds

As in the case of tori we start by fixing the representation of connection (3.4). In addition to representation of the torus generators (3.5) and (3.7) now we need to represent the $\mathbb{Z}_2$ generator W. As in the case of U_i's a generic representation of W

has the form $W = W^{\rm st} w$ where $W^{\rm st}$ is some standard representation satisfying

$$W^{\rm st}\nabla_i W^{\rm st} = -\nabla_i\,, \qquad W^{\rm st}U_i^{\rm st}W^{\rm st} = (U_i^{\rm st})^{-1}\,, \quad (W^{\rm st})^2 = 1$$

and w is a $N \times N$ matrix that obeys the relations

$$w u_i w = u_i^{-1}\,, \qquad w^2 = 1\,. \tag{5.1}$$

It is easy to check that one can take

$$W^{\rm st}:\; f_i(x) \longmapsto f_i(-x)\,,$$

where $f_i(x) \in E \cong L_2(\mathbb{R}^d)\otimes\mathbb{C}^N$ $(i = 1,\ldots,N)$. As before consider the decomposition of $\mathbb{C}^N$ into irreducible representations of the algebra (3.7) given by (3.17). The operator w maps $E_\Lambda \to E_{\Lambda^{-1}}$ where $\Lambda^{-1} \equiv (\lambda_1^{-1},\ldots,\lambda_d^{-1})$. Thus, the space $\mathbb{C}^N$ carries a representation of the algebra specified by (3.7) and (5.1) only if the set of $\{\Lambda_i\,, i = 1,\ldots,k\}$ in (3.17) is invariant under the inversion $\Lambda_i \mapsto \Lambda_i^{-1}$. We see that decomposition (3.17) can contain summands in the form of couples $\tilde{E}_\Lambda \equiv E_\Lambda \oplus E_{\Lambda^{-1}}$ and possible exceptional representations E_{Λ_ϵ} with $\Lambda_\epsilon = (\epsilon_1,\ldots,\epsilon_d)$, $\epsilon_i = \pm 1$, i.e. we have

$$\mathbb{C}^N = \left(\tilde{E}_{\Lambda_1}\oplus\cdots\oplus\tilde{E}_{\Lambda_r}\right)\bigoplus_{\epsilon} E_{\Lambda_\epsilon}^{\tau(\epsilon)}\,, \tag{5.2}$$

where $\tau(\epsilon)$ are non-negative integers. This decomposition can be chosen in such a way that the matrix w has a block diagonal form with r blocks $w_i : \tilde{E}_{\Lambda_i} \to \tilde{E}_{\Lambda_i}$ of the form

$$w_i = \begin{pmatrix} 0 & \tilde{w}_i \\ \tilde{w}_i^{-1} & 0 \end{pmatrix},$$

where $\tilde{w}_i : E_{\Lambda_i} \to E_{\Lambda_i^{-1}}$ and a number of blocks $w_\epsilon : E_{\Lambda_\epsilon} \to E_{\Lambda_\epsilon}$.

Let us first look at the components specified by pairs $(E_{\Lambda_\epsilon}, w_\epsilon)$. It follows from the irreducibility of the representation E_{Λ_ϵ} and the fact that $w_\epsilon^2 = 1$ that w_ϵ is defined up to an overall sign. Let $w_\epsilon^{(0)}$ be a standard choice of w_ϵ. Denote by $F_\epsilon^\pm$ the representation of the algebra (3.7) and (5.1) specified by $(E_{\Lambda_\epsilon}, \pm w_\epsilon^{(0)})$.

As for the blocks $(\tilde{E}_{\Lambda_i}, w_i)$ we first note that due to the irreducibility of E_{Λ_i} and $E_{\Lambda_i^{-1}}$ the operator $\tilde{w}_i$ is defined up to a constant factor. This constant factor can be gauged away by conjugating w_i with a suitable rescaling transformation

$$\begin{pmatrix} \mathbf{1}\cdot\mu_1 & 0 \\ 0 & \mathbf{1}\cdot\mu_2 \end{pmatrix},$$

where μ_1, μ_2 are complex numbers of absolute value 1 (these rescalings are allowed gauge transformations). Thus, we get a single representation of the algebra specified by (u_i, w) that we denote F_{Λ_i}. This representation is irreducible except for the cases when $\Lambda_i = \Lambda_\epsilon$ for some ϵ. Then we have $\tilde{w}_i = \pm w_\epsilon^{(0)}$ and one can readily check that $F_{\Lambda_\epsilon} \cong F_\epsilon^+ \oplus F_\epsilon^-$. This permits one to decompose $\mathbb{C}^N$ into the components

$$\mathbb{C}^N = (F_{\Lambda_1} \oplus \cdots \oplus F_{\Lambda_r}) \bigoplus_\epsilon (F_\epsilon^{\eta_\epsilon})^{\tau(\epsilon)}, \tag{5.3}$$

where $\eta_\epsilon = \pm$, $\tau(\epsilon)$ are non-negative integers specifying the multiplicities with which the corresponding representations enter the decomposition. The "representation I" and "representation II" of paper [18] are particular cases of the two types of elementary representations we have at hand: F_Λ, $F_\epsilon^{\eta_\epsilon}$. Note that the set of numbers Λ_i, r, $\tau(\epsilon)$ is uniquely determined by the given module E over $B_\theta = \mathbb{T}_\theta \rtimes \mathbb{Z}_2$ and by an equivariant constant curvature connection ∇_i on it. The residual gauge transformations preserving the decomposition (5.3) can be represented as compositions of transpositions acting inside each F_{Λ_i} block and sending $\Lambda_i \to \Lambda_i^{-1}$ and permutations of different F_{Λ_i} blocks. Thus, the moduli space of equivariant constant curvature connections is isomorphic to $(\mathbb{T}^d/\mathbb{Z}_2)^r/S_r$ where r is some integer. Using the relation

$$\text{g.c.d.}(\mu(E)) = 2r + \sum_\epsilon \tau(\epsilon)$$

one can find an estimate from above on r: $r \leq [\text{g.c.d.}(\mu)/2]$.

Let us comment here briefly on the invariance of the above results under Morita equivalence. It follows from the definition that the mapping of modules and connections induced by (complete) Morita equivalence relation preserves the gauge equivalence relation and maps equivariant constant curvature connections into equivariant constant curvature connections (see appendix A). Hence, the moduli spaces over the modules related by Morita equivalence are isomorphic. In particular the dimension r of the moduli space is preserved under Morita equivalences. Since the number g.c.d.$(\mu(E))$ is also preserved we see that $\sum_\epsilon \tau(\epsilon) = \text{g.c.d.}(\mu(E)) - 2r$ is preserved under Morita equivalence. In fact the set of pairs $(\tau(\epsilon), \eta_\epsilon)$ can only get permuted under Morita equivalence transformations. One can show that for modules admitting a constant curvature connection the set of parameters $\{(\tau(\epsilon), \eta_\epsilon)\}$ is in a one-to-one correspondence with additional topological numbers characterizing the module (see [21]).

6. $\mathbb{Z}_4$ orbifolds

In this section we will consider the case of toroidal $\mathbb{Z}_4$ orbifolds. For definiteness let us concentrate on the four-dimensional case. It is not hard to extend the results we obtain to other even-dimensional $\mathbb{Z}_4$ toroidal orbifolds. For $d = 4$ without loss of generality one can define a $\mathbb{Z}_4$ action on $\mathbb{R}^4$ by $\rho : (x, y) \mapsto (-y, x)$. Here we assume that $\mathbb{R}^4$ has a product structure $\mathbb{R}^2 \times \mathbb{R}^2$ and x, y belong to the first and second $\mathbb{R}^2$ factor, respectively. This action preserves the orthogonal lattice $D \cong \mathbb{Z}^4 \subset \mathbb{R}^4$ and

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thus descends to an action on the torus $\mathbb{R}^4/D$. We can consider a non-commutative four-torus $\mathbb{T}_\theta$ constructed by means of lattice D and an antisymmetric two-form θ_{ij} that is assumed to be decomposed into a 2×2 block form relative to the $\mathbb{R}^2 \times \mathbb{R}^2$ product structure. One can easily see that such a form is invariant with respect to the above defined $\mathbb{Z}_4$ action. Thus, we can consider a non-commutative toroidal orbifold $\mathbb{T}_\theta \rtimes_\rho \mathbb{Z}_4$.

Let E be a projective module over this orbifold and let Y denotes a representation of the $\mathbb{Z}_4$ generator. Applying the general construction of section 4 we arrive at the decomposition (3.17) and a representation of a matrix algebra generated by u_i and y, satisfying (3.7) and

$$y^4 = 1\,, \qquad y^{-1} u_{\mathbf{n}} y = u_{\rho^{-1}\mathbf{n}}\,. \tag{6.1}$$

The total representation space $\mathbb{C}^N$ splits into irreducible representations of this matrix algebra. To describe those we first classify the orbits of the $\mathbb{Z}_4$ action ρ^* on the dual torus $\tilde{\mathbb{T}}^4 \cong \mathbb{R}^{*4}/\tilde{D}^*$. For a generic point of $\tilde{\mathbb{T}}^4$ the orbit consists of 4 distinct points. There are 16 exceptional points. Those include the four fixed points: $(0,0;0,0)$, $(0,1/2;0,1/2)$, $(1/2,0;1/2;0)$, and $(1/2,1/2,1/2,1/2)$, and 12 points whose orbit is of length 2. The last ones have coordinates 0 or 1/2 and they complete the $\mathbb{Z}_4$ action fixed points written above to the whole set of 16 $\mathbb{Z}_2$ action fixed points (the $\mathbb{Z}_2$ action is specified by ρ^2). Hence, we have 4 orbits of order 1 and 6 orbits of order 2. Denote the fixed points Λ_ϵ, $\epsilon = 1, \ldots, 4$ and the pairs of points whose orbits are of order 2 by $(\Lambda'_\nu, \Lambda''_\nu)$, $\nu = 1, \ldots, 6$.

Let us first consider the representations corresponding to generic points. Denote

$$\tilde{E}_\Lambda = E_\Lambda \oplus E_{\rho^*\Lambda} \oplus E_{(\rho^*)^2\Lambda} \oplus E_{(\rho^*)^3\Lambda}\,.$$

Relative to this decomposition the generator y can be written in a block form as

$$y = \begin{pmatrix} 0 & y_{12} & 0 & 0 \\ 0 & 0 & y_{23} & 0 \\ 0 & 0 & 0 & y_{34} \\ y_{41} & 0 & 0 & 0 \end{pmatrix},$$

where $y_{ij} : E_{(\rho^*)^{i-1}\Lambda} \to E_{(\rho^*)^i\Lambda}$. It follows from irreducibility of representations E_Λ that the blocks y_{ij} are determined uniquely up to constant factors. These constant factors can be gauged away by gauge transformations that multiply each of the $E_{(\rho^*)^n\Lambda}$ components of $\tilde{E}_\Lambda$ by a constant. Thus, we obtain a representation F_Λ of the matrix algebra (3.7) and (6.1) (we hope that using the same notation as the one used in the previous section when studying the $\mathbb{Z}_2$ case will not cause any confusion).

Next let us look at representations labeled by pairs of exceptional points $(\Lambda'_\nu, \Lambda''_\nu)$. The representation of rational torus is $\hat{E}_{\Lambda_\nu} = E_{\Lambda'_\nu} \oplus E_{\Lambda''_\nu}$. A general form of the generator y is

$$y = \begin{pmatrix} 0 & \mu_1 \cdot y_1 \\ \mu_2 \cdot y_2 & 0 \end{pmatrix},$$

where $y_1 : E_{\Lambda''_\nu} \to E_{\Lambda'_\nu}$, $y_2 : E_{\Lambda'_\nu} \to E_{\Lambda''_\nu}$ are fixed and μ_1, μ_2 are constants satisfying $(\mu_1\mu_2)^2 = 1$. Using gauge transformations one can bring y to one of the two forms specified by $\mu_1 = \mu_2 = 1$ and $\mu_1 = -\mu_2 = 1$. Therefore, we have two inequivalent representations of the algebra (3.7) and (6.1) denoted $G_\nu^\pm$ with $\pm$ standing for the sign of the product $\mu_1\mu_2$.

Finally, consider the fixed points Λ_ϵ. Since each E_{Λ_ϵ} is an irreducible representation of a rational torus the operator y acting on it is defined uniquely up to a multiplication by a 4th root of unity $\xi_k = \exp(2\pi i k/4)$, $k = 0, 1, 2, 3$. Thus, for every fixed point we have 4 different representations F_ϵ^k, $k = 0, \ldots, 3$.

The generic representation F_Λ is irreducible unless Λ hits one of the exceptional points. If it hits a fixed point the representation splits as $F_{\Lambda_\epsilon} \cong \oplus_{k=0}^3 F_\epsilon^k$. In this case the representation of $\mathbb{Z}_4$ is a tensor product of a regular representation with a representation that acts on a single copy of E_{Λ_ϵ}. If Λ coincides with one of Λ'_ν or Λ''_ν the corresponding representation splits as $F_{\Lambda'_\nu} \cong F_{\Lambda''_\nu} \cong G_\nu^+ \oplus G_\nu^-$. Using these equivalences we can decompose a general representation as

$$\mathbb{C}^N = (F_{\Lambda_1} \oplus \cdots \oplus F_{\Lambda_r}) \bigoplus_\nu (G_\nu^{\zeta_\nu})^{s(\nu)} \bigoplus_\epsilon \left((F_\epsilon^{\eta_\epsilon^1})^{\tau^1(\epsilon)} \oplus (F_\epsilon^{\eta_\epsilon^2})^{\tau^2(\epsilon)} \oplus (F_\epsilon^{\eta_\epsilon^3})^{\tau^3(\epsilon)} \right), \quad (6.2)$$

where $\zeta_\nu = \pm$; η_ϵ^i, $i = 1, 2, 3$ are a triple of distinct integers from 0 to 4; $\tau^i(\epsilon)$ and $s(\nu)$ are non-negative integers standing for the multiplicities of modules. The numbers r, $\tau^i(\epsilon)$, $s(\nu)$, $\zeta_\nu = \pm$, η_ϵ^i are uniquely determined by a given module. In is straightforward to generalize considerations of the previous section to show that the moduli space of equivariant constant curvature connections is isomorphic to $(\tilde{\mathbb{T}}^d/\mathbb{Z}_4)^r/S_r$ where r is some integer such that $r \leq [\text{g.c.d.}(\mu)/4]$. The last restriction follows from the relation

$$\text{g.c.d.}(\mu) = 4r + 2\left(\sum_\nu s(\nu)\right) + \sum_\epsilon \sum_{i=1}^3 \tau^i(\epsilon)\,.$$

7. Coulomb branches of the moduli space

Once we described moduli space of constant curvature connections it is not hard to add scalar fields to the discussion. Let us first consider the case of tori. The equations for scalars that we have are

$$[\nabla_i, X_I] = 0\,, \qquad [X_I, X_J] = 0\,, \qquad [U_i, X_I] = 0\,. \quad (7.1)$$

Here $I = 1, \ldots, 9-d$. For the fixed representation of ∇_i (3.4) the first equation in (7.1) implies that X_I are constant $N \times N$ matrices. It follows then from the last two equations that the matrices X_I, u_i can be simultaneously brought to the form when X_I's are diagonal and u_i's are block diagonal corresponding to the decomposition (3.17). Moreover the X_I's are constant on each irreducible component E_{Λ_i}.

Quotienting over S_{N_0} residual gauge transformations gives us Coulomb branches isomorphic to $(\mathbb{R}^{9-d})^{N_0}/S_{N_0}$ that matches with a dual (Morita equivalent) description of this system as a system of $N_0 = \text{g.c.d.}(\mu(E))$ D0-branes.

For the case of a $\mathbb{Z}_2$ orbifold in addition to eqs. (7.1) we have the condition

$$[W, X_I] = 0\,. \tag{7.2}$$

This implies that the matrices X_I commute with the matrix w. Thus, the matrices u_i, w, and X_I's can be simultaneously brought to the form that corresponds to the decomposition (5.2) and X_I are in a block diagonal form with r blocks $x_I^{(i)} : F_{\Lambda_i,\eta_i} \to F_{\Lambda_i,\eta_i}$, $i = 1,\ldots,r$ and blocks of the form $x_I^{\epsilon} : F_{\epsilon} \to F_{\epsilon}$. The last ones are necessarily constants due to irreducibility of E_{Λ_i}. If none of the points Λ_i coincides with one of the fixed points Λ_ϵ we obtain Coulomb branches of the form

$$\left(\mathbb{R}^{(9-d)}\right)^r/S_r \times \prod_{\epsilon}\left((\mathbb{R}^{(9-d)})^{\tau(\epsilon)}/S_{\tau(\epsilon)}\right)\,.$$

If $\Lambda_{i_k} = \Lambda_\epsilon$ for some subset of indices i_k, $k = 1,\ldots,p$ and for some ϵ, each representation $F_{\lambda_{i_k},\eta}$ breaks into a direct sum of two representations F_ϵ and F_ϵ^- and instead of the factor $(\mathbb{R}^{(9-d)})^r/S_r \times ((\mathbb{R}^{(9-d)})^{\tau(\epsilon)}/S_{\tau(\epsilon)})$ we get a factor of

$$\left(\mathbb{R}^{(9-d)}\right)^{r-p}/S_{r-p} \times \left(\mathbb{R}^{(9-d)}\right)^{p}/S_p \times \left(\mathbb{R}^{(9-d)}\right)^{\tau(\epsilon)+p}/S_{\tau(\epsilon)+p}\,. \tag{7.3}$$

This picture has an interpretation suggested in [18] in terms of splitting of a D0 particle into a membrane-antimembrane pair that occurs once the D0 particle hits a fixed point. In terms of this interpretation (7.3) corresponds to p membranes (or antimembranes depending on the value of η_ϵ) and $p+\tau(\epsilon)$ antimembranes (membranes) sitting at the fixed point Λ_ϵ.

Now let us look at the $\mathbb{Z}_4$ case. Again we have a set of $9-d$ $N \times N$ matrices X_I. These matrices commute between themselves and with the matrix y. This leads us to a block decomposition of each of the X_I relative to the decomposition (6.2) in which X_I is constant on every irreducible representation of the algebra generated by u_i, y. Thus, for a generic point in the moduli space of constant curvature connection the Coulomb branch is

$$\left(\mathbb{R}^{(9-d)}\right)^r/S_r \times \prod_{\nu}\left(\mathbb{R}^{(9-d)}\right)^{s(\nu)}/S_{s(\nu)}\prod_{\epsilon}\prod_{i=1}^{3}\left(\mathbb{R}^{(9-d)}\right)^{\tau^i(\epsilon)}/S_{\tau^i(\epsilon)}\,.$$

If one of Λ_i in (6.2) coincides with one of exceptional points Λ'_ν, Λ''_ν or Λ_ϵ representations F_{Λ_i} split accordingly as $F_{\Lambda'_\nu} \cong F_{\Lambda''_\nu} \cong G_\nu^+ \oplus G_\nu^-$, or $F_{\Lambda_\epsilon} \cong \oplus_{k=0}^3 F_\epsilon^k$. In that case the block decomposition of X_I is different and the answer is best described using brane terminology. In general the moduli space coincides with the one describing a system consisting of a number of free D0 particles, single membranes that are stuck at fixed points Λ_ϵ and pairs of membranes sitting at points $(\Lambda'_\nu, \Lambda''_\nu)$. It seems to be natural to identify parameters ζ_ν, η_ϵ specifying different types of membranes with a discrete B-field flux carried by them.

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A. Morita equivalence of toroidal orbifolds

In this appendix we will show how Morita equivalences known for non-commutative tori can be extended to non-commutative toroidal orbifolds. We will give a general construction for arbitrary orbifold group G and then specialize to the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ cases.

We start by reminding the reader some basic definitions concerning Morita equivalence (see [5] for details). Let A and $\hat{A}$ be two involutive associative algebras. A $(A, \hat{A})$-bimodule P is said to establish a Morita equivalence between A and $\hat{A}$. This means that the projective bimodule P obeys the conditions

$$\bar{P} \otimes_A P \cong \hat{A}\,, \qquad P \otimes_{\hat{A}} \bar{P} \cong A\,, \tag{A.1}$$

where $\bar{P}$ is a $(\hat{A}, A)$-bimodule that is complex conjugated to P that means that $\bar{P}$ consists of elements of P and multiplications are defined as $\hat{a}(e) := (e)\hat{a}^*$, $(e)a := a^*(e)$ where multiplications on the right hand sides are those defined for the bimodule P. The algebras A and $\hat{A}$ are said to be Morita equivalent if such P satisfying (A.1) exists. The bimodule P determines a one to one correspondence between A-modules and $\hat{A}$-modules by the rule

$$\hat{E} = E \otimes_A P\,, \quad E = \hat{E} \otimes_{\hat{A}} \bar{P}\,.$$

For the case of non-commutative tori one can define a notion of complete Morita equivalence ([5]) that allows one to transport connections between modules E and $\hat{E}$. We say that $(\mathbb{T}_\theta, A_{\hat{\theta}})$ Morita equivalence bimodule P establishes a complete Morita equivalence if it is endowed with operators ∇_i^P that determine a constant curvature connection simultaneously with respect to $\mathbb{T}_\theta$ and $\mathbb{T}_{\hat{\theta}}$, i.e. satisfy

$$\begin{gathered} \nabla_i^P(ae) = a\nabla_i^P e + (\delta_i a)e\,, \qquad \nabla_i^P(e\hat{a}) = (\nabla_i^P e)\hat{a} + e\hat{\delta}_i\hat{a}\,, \\ \left[\nabla_i^P, \nabla_j^P\right] = \sigma_{ij} \cdot \mathbf{1}\,. \end{gathered} \tag{A.2}$$

Here δ_i and $\hat{\delta}_j$ are standard derivations on $\mathbb{T}_\theta$ and $\mathbb{T}_{\hat{\theta}}$, respectively. Sometimes for brevity we will omit the word Morita in the term (complete) Morita equivalence bimodule. If P is a complete $(\mathbb{T}_\theta, \mathbb{T}_{\hat{\theta}})$ equivalence bimodule then there exists a correspondence between connections on E and connections on $\hat{E}$. An operator $\nabla_i \otimes 1 + 1 \otimes \nabla_i^P$ on $E \otimes_{\mathrm{C}} P$ descends to a connection $\hat{\nabla}_\alpha$ on $\hat{E} = E \otimes_{\mathbb{T}_\theta} P$. The curvatures of $\hat{\nabla}_i$ and ∇_i are connected by the formula $F_{ij}^{\hat{\nabla}} = \hat{F}_{ij}^{\nabla} + \mathbf{1}\sigma_{ij}$.

Given a $(\mathbb{T}_\theta, \mathbb{T}_{\hat\theta})$ equivalence bimodule P there is a possibility of extension of the equivalence relation that it defines to modules over orbifolds $\mathbb{T}_\theta \rtimes_R G$, $\mathbb{T}_{\hat\theta} \rtimes_R G$. We will first describe a general construction and then discuss for which bimodules P it exists.

Assume that P is equipped with a set of operators $W^P(g)$, $g \in G$ satisfying

$$\begin{aligned} &W^P(g)W^P(h) = W(hg)\,, \\ &(W^P(g))^{-1}U_{\mathbf{n}}W^P(g) = U_{R\mathbf{n}}\,, \qquad \left(W^P(g)\right)^{-1}\hat U_{\mathbf{n}}W^P(g) = \hat U_{R\mathbf{n}}\,, \end{aligned} \tag{A.3}$$

where $U_{\mathbf{n}}$ and $\hat U_{\mathbf{n}}$ stand for actions of $\mathbb{T}_\theta$ and $\mathbb{T}_{\hat\theta}$ generators, respectively. The first equation in (A.3) means that operators $W^P(g)$ give a right action of the group G on E that is just a choice of conventions. If F is a right module over $\mathbb{T}_\theta \rtimes_R G$ specified by a $\mathbb{T}_\theta$-module E and operators $W(g)$ acting on it one defines a right $\mathbb{T}_{\hat\theta} \rtimes_R G$ module $\hat F$ as a $\mathbb{T}_{\hat\theta}$-module $\hat E = E \times_{\mathbb{T}_\theta} P$ equipped with operators $\hat W(g) := W(g) \otimes W^P(g)$. Analogously one defines a mapping in the opposite direction.

Given a pair $(P, \{W^P(g); g \in G\})$ as above one can construct a true $(\mathbb{T}_\theta \rtimes_R G, \mathbb{T}_{\hat\theta} \rtimes_R G)$ Morita equivalence bimodule in the sense of general definition given at the beginning of this section. The construction goes as follows. To shorten notations denote $B_\theta = \mathbb{T}_\theta \rtimes_R G$, $B_{\hat\theta} = \mathbb{T}_{\hat\theta} \rtimes_R G$. Elements of the $(B_\theta, B_{\hat\theta})$ bimodule that we denote Q are pairs $p \otimes_{\mathbb{C}} \sum_g c(g)g$ where $p \in P$ and $\sum_g c(g)g$ are formal linear combinations of elements of the group G with complex coefficients. Multiplication by complex numbers on Q is defined in the obvious way. We further define left and right actions of generators of B_θ and $B_{\hat\theta}$ as

$$\begin{aligned} &U_{\mathbf{n}}(p \otimes g) = (U_{R^{-1}(g)\mathbf{n}}p) \otimes g\,, \qquad W(h)(p \otimes g) = p \otimes (hg)\,, \\ &(p \otimes g)\hat U_{\mathbf{n}} = (p\hat U_{\mathbf{n}}) \otimes g\,, \qquad (p \otimes g)\hat W(h) = \left(pW^P(h)\right) \otimes (gh)\,. \end{aligned} \tag{A.4}$$

One can check that Q satisfies the defining properties of Morita equivalence bimodule, i.e.

$$\bar Q \otimes_{B_\theta} Q \cong B_{\hat\theta}\,, \qquad Q \otimes_{B_{\hat\theta}} \bar Q \cong B_\theta\,.$$

Namely, let fix us the isomorphisms $I : \bar P \otimes_{\mathbb{T}_\theta} P \to \mathbb{T}_{\hat\theta}$, $\bar I : P \otimes_{\mathbb{T}_{\hat\theta}} \bar P \to A_\theta$, then one can define a mapping $J : \bar Q \otimes_{B_\theta} Q \to B_{\hat\theta}$ as

$$J\Big((\bar p \otimes \bar g) \otimes_{B_\theta} (p \otimes g)\Big) = I\Big(\bar p \otimes \left(pW^P(g^{-1}\bar g)\right)\Big) \cdot (\bar g^{-1}g)\,.$$

It is easy to check that J is an isomorphism of the corresponding $(B_{\hat\theta}, B_{\hat\theta})$ bimodules. An isomorphism $\bar J : Q \otimes_{B_{\hat\theta}} \bar Q \to B_\theta$ can be constructed in a similar way.

We further need to construct a correspondence between equivariant connections on B_θ and $B_{\hat\theta}$ modules F, $\hat F$. We call an $\mathbb{T}_\theta$-connection ∇_i defined on a B_θ-module E equivariant if it satisfies a constraint

$$W^{-1}(g)\nabla_i W(g) = R_i^j(g)\nabla_j\,. \tag{A.5}$$

Now let us assume that we are given a triple $(P, \nabla^P, W^P(g))$ where P is a $(\mathbb{T}_\theta, \mathbb{T}_{\hat{\theta}})$ Morita equivalence bimodule, operators $W^P(g)$ acting on P satisfy (A.3), ∇_i^P satisfies (A.2) and an additional equivariance constraint (A.5). The couple (P, ∇^P) establishes a complete Morita equivalence of the algebras $\mathbb{T}_\theta$. It is simple to check that due to condition (A.5) a mapping of $\mathbb{T}_\theta$- connections that is defined by this couple preserves the equivariance condition. We will say that a triple $(P, \nabla^P, W^P(g))$ specifies a $(B_\theta, B_{\hat{\theta}})$ complete equivalence. If a triple $(P, \nabla_\alpha^P, W^P(g))$ specifies a $(B_\theta, B_{\theta'})$ complete equivalence and a triple $(P', \nabla_\alpha^{P'}, W^{P'}(g))$ specifies a $(B_{\theta'}, B_{\theta''})$ complete equivalence, then the tensor product $P'' = P \otimes_{B_{\theta'}} P'$ along with the appropriate connection $\nabla_\alpha^{P''}$ and involution $W^{P''}$ determines $(B_\theta, B_{\theta''})$ complete equivalence. This means that we can consider a groupoid of equivalences.

Let us discuss now the existence of triples $(P, \nabla^P, W^P(g))$. First we would like to note that not every (complete) Morita equivalence of tori can be lifted to orbifolds. There is an obvious constraint that the matrix θ should stay invariant under the orbifold group action. Morita equivalence of $\mathbb{T}_\theta$ algebras is governed by the group $\mathrm{SO}(d, d|\mathbb{Z})$. An element $g \in \mathrm{SO}(d, d|\mathbb{Z})$ can be represented by a $2d \times 2d$ block matrix

$$g = \begin{pmatrix} M & N \\ R & S \end{pmatrix}$$

and the action on θ is given by a fractional transformation

$$g: \ \theta \longmapsto (M\theta + N)(R\theta + S)^{-1} . \tag{A.6}$$

The orbifold group G naturally acts on θ as

$$h: \ \theta \longmapsto R^t(h)\theta R(h) , \quad h \in G .$$

The condition that two actions commute specifies a certain subgroup of $\mathrm{SO}(d, d|\mathbb{Z})$ and can be formalized as follows. We can embed both groups into $\mathrm{O}(d, d|\mathbb{R})$. For the first group the embedding is obvious, for the orbifold group we embed an element specified by the matrix $R(g)$ into

$$R(g) \longmapsto \begin{pmatrix} R^t(g) & 0 \\ 0 & R^t(g^{-1}) \end{pmatrix} .$$

Now the commutation condition has a precise meaning. We proceed to the construction of $(P, \nabla^P, W^P(g))$ triples. The group $\mathrm{SO}(d, d|\mathbb{Z})$ is generated by the following transformations. The first type of transformations is a subgroup $\mathrm{SL}(d, \mathbb{Z})$ of modular transformations. The second type consists of shifts $\theta \mapsto \theta + N$ where N is an arbitrary antisymmetric matrix with integer entries. To generate the whole $\mathrm{SO}(d, d|\mathbb{Z})$ one has to add a "flip" transformation σ that inverts a 2×2 non-degenerate block in the matrix θ. Namely, without loss of generality we can assume that θ has a block form

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} ,$$

where θ_{11} is a $2p \times 2p$ non-degenerate matrix. Then a flip σ_{2p} sends θ into

$$\sigma_2(\theta) = \begin{pmatrix} \theta_{11}^{-1} & -\theta_{11}^{-1}\theta_{12} \\ \theta_{21}\theta_{11}^{-1} & \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12} \end{pmatrix} . \tag{A.7}$$

One can check that modular transformations, shifts and the flip σ_2 generate the whole $\mathrm{SO}(d, d|\mathbb{Z})$. For any such transformation that commutes with G one can construct a triple $(P, \nabla^P, W^P(g))$. Let $A \in \mathrm{SL}(d, \mathbb{Z})$ and A commute with G. The corresponding $(\mathbb{T}_\theta, \mathbb{T}_{A^t\theta A}$ equivalence bimodule consists of elements $a \in \mathbb{T}_\theta$ and the actions of generators are defined as

$$U^l_{\mathbf{n}}(a) = U_{\mathbf{n}}a\,, \qquad (a)U^r_{\mathbf{n}} = aU_{A\mathbf{n}}\,, \tag{A.8}$$

where to avoid confusion we denoted by $U_{\mathbf{n}}$ elements of $\mathbb{T}_\theta$, and by $U^l_{\mathbf{n}}$, $U^r_{\mathbf{n}}$ left and right actions of the corresponding tori. This bimodule can be completed to a triple by adding the following operators ∇^P_i and $W^P(g)$

$$W^P(g)(a) = \rho_g(a)\,, \qquad \nabla^P_i(a) = \delta_i(a)\,, \tag{A.9}$$

where ρ_g stands for automorphisms of algebra $\mathbb{T}_\theta$ induced by the action $R(g)$ on the lattice.

Let $N = (N^{ij})$ be an antisymmetric G-invariant matrix with integer entries. The corresponding $(\mathbb{T}_\theta, \mathbb{T}_{\theta+N})$ equivalence bimodule consists of elements $a \in \mathbb{T}_\theta$ with tori actions defined as

$$U^l_{\mathbf{n}}(a) = U_{\mathbf{n}}a\,, \qquad (a)U^r_{\mathbf{n}} = aU_{\mathbf{n}}(-1)^{\sum_{i<j} n_i N^{ij} n_j}\,. \tag{A.10}$$

The operators ∇^P_i, $W^P(g)$ are the same as in (A.8) and (A.9).

Finally we need to define a triple corresponding to a G-invariant flip (A.7). Although it is not hard if only somewhat lengthy to describe triples corresponding to generic G-invariant flips of the type σ_2 we will confine ourselves to the "total" flip that inverts the whole matrix θ (provided the last one is invertible). The formulas for that case are most succinct and elegant. Besides this case alone will be sufficient for all our future needs. The operators ∇^P_j should satisfy

$$\left[\nabla^P_j, \nabla^P_k\right] = -2\pi i\theta_{jk} \cdot \mathbf{1}\,. \tag{A.11}$$

Since θ is non degenerate the last relation defines a Heisenberg algebra. The $(\mathbb{T}_\theta,\mathbb{T}_{\theta^{-1}})$ bimodule is modeled on $L_2(\mathbb{R}^{d/2})$ space that is assumed to carry an irreducible representation of the Heisenberg algebra (A.11). The tori actions are defined to be given by the following operators on $L_2(\mathbb{R}^{d/2})$

$$U^l_j = \exp(\nabla^P_j)\,, \qquad U^r_j = \exp\left((\theta^{-1})^{jk}\nabla^P_k\right). \tag{A.12}$$

Now the matrix θ is assumed to be G-invariant. This assumption implies that the transformation $\nabla_j \mapsto R^k_j\nabla^P_k$ is an isomorphism of the Heisenberg algebra (A.11).

There exists then a set of unitary operators $W^P(g)$ satisfying (A.5). It follows from (A.5) and (A.12) that the second equation in (A.3) holds. As for the first equation in general it is satisfied up to a phase factor, i.e. $W^P(g)$ determine a projective representation of G. This possibility can be easily embedded into a general theory discussed above. However we will assume that the first equation in (A.3) is satisfied precisely which is always the case for cyclic orbifolds. This finishes the construction of the corresponding triple.

Now we specialize to the cases of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds. We will show that by use of Morita equivalence any module admitting a constant curvature equivariant connection can be mapped to a module with a non-degenerate curvature tensor (the dimension d is assumed to be even). First consider $\mathbb{Z}_2$ orbifolds. In this case the mapping $x \mapsto -x$ always preserves the lattice D and any antisymmetric two-form θ on it. Moreover any transformation from the group $\mathrm{SO}(d,d|\mathbb{Z})$ commutes with the $\mathbb{Z}_2$ transformation. In general under transformation (A.6) the curvature tensor $f = (f_{ij})$ transforms as

$$\tilde{f} = AfA^t + RA^t\,, \tag{A.13}$$

where $A = R\theta + S$. The matrix $\tilde{f}$ is non degenerate if the matrix $f + A^{-1}R$ is non degenerate. One can check that there exist $\mathrm{SO}(d,d|\mathbb{Z})$ transformations such that the matrix $A^{-1}R$ is invertible and its matrix norm can be made arbitrarily large. It follows from this that $f + A^{-1}R$ can be made non degenerate.

In the $\mathbb{Z}_4$ case we can choose a basis in which a $\mathbb{Z}_4$ generator has a form

$$\begin{pmatrix} 0 & 1_{2\times 2} \\ -1_{2\times 2} & 0 \end{pmatrix},$$

where for simplicity we set $d = 4$. An arbitrary $\mathbb{Z}_4$ invariant matrix then has a form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where A, B are arbitrary 2×2 matrices. In particular we see that we have a big supply of invariant matrices with integral entries. This allows one to repeat the argument made above for the $\mathbb{Z}_2$ case.

References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[3] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

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[4] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[5] A. Schwarz, *Morita equivalence and duality*, *Nucl. Phys.* **B 534** (1998) 720 [hep-th/9805034].

[6] P.-M. Ho and Y.-S. Wu, *Noncommutative gauge theories in matrix theory*, *Phys. Rev.* **D 58** (1998) 066003 [hep-th/9801147].

[7] P.-M. Ho and Y.-S. Wu, *Matrix compactification on orientifolds*, *Phys. Rev.* **D 60** (1999) 026002 [hep-th/9812143].

[8] Y.-K.E. Cheung and M. Krogh, *Noncommutative geometry from 0-branes in a background B-field*, *Nucl. Phys.* **B 528** (1998) 185 [hep-th/9803031].

[9] T. Kawano and K. Okuyama, *Matrix theory on noncommutative torus*, *Phys. Lett.* **B 433** (1998) 29 [hep-th/9803044].

[10] C. Hofman and E. Verlinde, *Gauge bundles and Born-Infeld on the noncommutative torus*, *Nucl. Phys.* **B 547** (1999) 157 [hep-th/9810219].

[11] A. Konechny and A. Schwarz, *BPS states on noncommutative tori and duality*, *Nucl. Phys.* **B 550** (1999) 561 [hep-th/9811159].

[12] A. Konechny and A. Schwarz, *Supersymmetry algebra and BPS states of super Yang-Mills theories on noncommutative tori*, *Phys. Lett.* **B 453** (1999) 23 [hep-th/9901077].

[13] D. Brace, B. Morariu and B. Zumino, *Dualities of the matrix model from T-duality of the type-II string*, *Nucl. Phys.* **B 545** (1999) 192 [hep-th/9810099].

[14] C.-S. Chu and P.-M. Ho, *Constrained quantization of open string in background B field and noncommutative D-brane*, *Nucl. Phys.* **B 568** (2000) 447 [hep-th/9906192].

[15] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[16] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[17] Z. Guralnik and S. Ramgoolam, *Torons and D-brane bound states*, *Nucl. Phys.* **B 499** (1997) 241 [hep-th/9702099]; Z. Guralnik and S. Ramgoolam, *From 0-branes to torons*, *Nucl. Phys.* **B 521** (1998) 129 [hep-th/9708089].

[18] S. Ramgoolam and D. Waldram, *Zero branes on a compact orbifold*, *J. High Energy Phys.* **07** (1998) 009 [hep-th/9805191].

[19] B.R. Greene, C.I. Lazaroiu and P. Yi, *D-particles on $T^4/\mathbb{Z}_N$ orbifolds and their resolutions*, *Nucl. Phys.* **B 539** (1999) 135 [hep-th/9807040].

[20] M.R. Douglas, *D-branes and discrete torsion*, hep-th/9807235.

[21] A. Konechny and A. Schwarz, *Compactification of M(atrix) theory on noncommutative toroidal orbifolds*, hep-th/9912185 to appear in *Nucl. Phys.* **B**.

[22] G. Bertoldi, J.M. Isidro, M. Matone and P. Pasti, *Noncommutative riemann surfaces*, hep-th/0003131; G. Bertoldi, J.M. Isidro, M. Matone and P. Pasti, *The concept of a noncommutative riemann surface*, *Phys. Lett.* **B 484** (2000) 323 [hep-th/0003200].

[23] M.A. Rieffel, *Non-commutative tori. A case study of non-commutative differentiable manifolds*, *Contemporary Math.* **105** (1990) 191.

[24] A. Connes and M. Rieffel, *Yang-Mills for non-commutative two-tori*, in *Operator algebras and mathematical physics*, Iova City, Iova, 1985, pp. 237-266.

[25] M. Rieffel and A. Schwarz, *Morita equivalence of multidimensional noncommutative tori*, *Int. J. of Math.* **10, 2** (1999) 289, math.QA/9803057.

[26] A. Astashkevich and A. Schwarz , *Projective modules over noncommutative tori: classification of modules with constant curvature connection*, math.QA/9904139, to appear in *J. of Operator Theory.*

[27] M. Rieffel, *Projective modules over higher-dimensional non-commutative tori*, *Can. J. Math.* **XL, 2** (1988) 257.

[28] S.G. Walters, *Projective modules over the noncommutative sphere*, *J. London Math. Soc.* **(2) 51** (1995) 589; *Chern characters of Fourier modules*, to appear on *Can. J. Math* (1999) 39; *K-theory of noncommutative spheres arising from the Fourier automorphism*, preprint (1999), 38 pages.

[29] P. van Baal, *A simple construction of twist eating solutions*, *J. Math. Phys.* **27** (1986) 455.

[30] J. Igusa, *Theta functions*, Springer, 1972.